On the Resonances of a Dielectric Resonator of Very High Permittivity

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Abstract—**It is shown that the modes of a dielectric resonator are of two types: confined and nonconfined. Orthogonality properties and variational principles are derived for these modes, and their radiation pattern and quality factor are investigated. The material of the resonator is assumed lossless and of very high permittivity.**

I. INTRODUCTION

A NONMAGNETIC lossless body of dielectric constant $\epsilon_r = N^2$, excited by sinusoidal volume sources $\bar{J}$ is shown in Fig. 1. It is well known that the fields in and around the resonator peak to high values at certain (resonant) frequencies, and that the sharpness of the resonance increases with N, the index of refraction. Resonance phenomena also occur when the dielectric is immersed in an (incident) external field. Both cases are considered in detail in a companion article. In the present paper, we concentrate our attention on the properties of the resonant modes, and in particular, on their field patterns and quality factors.

Resonant modes are field structures which can exist in the absence of $\bar{J}$. It is apparent that their determination is a "coupled-regions" problem, as fields exist in V *and* V'. An exact solution for arbitrary N is possible for a few simple shapes, e.g., the sphere, for which separation of variables is applicable [1], [2]. For a resonator of arbitrary shape, general results are difficult to obtain. In order to make some progress, we shall introduce a simplifying feature, and assume that the permittivity of the dielectric approaches infinity. It should be immediately evident that the problem remains of the "coupled-regions" type, and that boundary surface S does not generally behave as a "magnetic wall." Consider Fig. 1, for example, and assume that S is a magnetic wall (or "open circuit"). This assumption implies that $\bar{H}_{\tan}$ vanishes along S. But $\bar{H}_{\tan}$ is, in the case of Fig. 1, the only source of the field outside S. It follows that the external field is zero. In consequence, H_n must also vanish along S. We conclude that S can behave as a magnetic wall only if $\bar{H}_{\tan} = 0$ implies $H_n = 0$ on S. Clearly, such a restricted situation can only exist for very special symmetries of the resonator and of its source distribution. We must therefore accept that $\bar{H}$ penetrates into the vacuum region, even for $N \to \infty$. In that case, however, the field remains confined in the immediate vicinity of the resonator. It is sometimes

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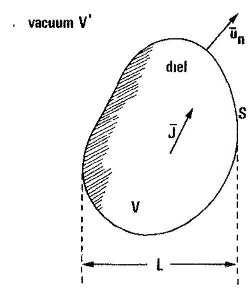

Fig. 1. Dielectric resonator with volume sources.

assumed that the fields decrease exponentially with distance, as in the case of a plane wave totally reflected from a dielectric–air interface. Richtmyer recognized early [3] that the exponential decrease could not hold for all directions, as it would entail zero radiated power and infinite Q. To avoid solving the coupled-regions problem, early investigators [4] assumed that S was a magnetic wall. It was pointed out [5], [6] that this assumption could give acceptable results for the higher order modes, but not for the lower order ones. A better approximation, leaving room for field penetration outside the resonator, was obtained for the pillbox resonator by assuming that the lateral surfaces are magnetic walls, but that the flat ends allow fields to leak out [7], [8]. Leakage is calculated by assuming that the resonator is enclosed in an infinite waveguide with magnetic walls. In the present paper, we attempt to free the analysis from these "ad hoc" assumptions, and to formulate the problem in a rigorous way.

II. GENERAL METHOD OF SOLUTION

A. *Expansion in Powers of 1/N*

In letting N approach infinity, two approaches are possible. The first is to keep the frequency constant and let the number of wavelengths in V grow without limit. The second is to concentrate on a given resonant mode, corresponding to a finite wavenumber $\boldsymbol{k}$ in the dielectric, and to see what happens to the fields as $N \to \infty$. During this limiting process, kL approaches an asymptotic value, finite and different from zero (L is a typical dimension of the resonator). This value is a characteristic of the mode. The wavenumber k_0 in vacuo ($k_0 = 2\pi/\lambda_0 = k/N = \omega/c_0$) approaches zero, together with the frequency, and the wavelength λ_0 approaches infinity. We shall follow the second limiting process, which is more realistic as dielectric resonators are normally dimensioned to resonate in one of their lowest modes. The mathematical solution proceeds by expanding the fields in a series:

$$\bar{E} = \bar{E}_0 + \bar{E}_1/N + \bar{E}_2/N^2 + \cdots$$
$$\bar{H} = \bar{H}_0 + \bar{H}_1/N + \bar{H}_2/N^2 + \cdots. \quad (1)$$

Expansions of the same kind have been introduced by Stevenson [9], who utilizes the wavenumber $k_0 = k/N$ as an expansion parameter. Stevenson's method has been applied to dielectric bodies, but under the assumption that their dimensions are small with respect to $\lambda_{\text{diel}} = \lambda_0/N$ [11]. This restriction obviously excludes the study of resonances, where λ_{diel} is of the order of the dimensions. Expressions (1) do not suffer from this limitation. We insert them in Maxwell's equations, and equate coefficients of equal powers of $1/N$ on both sides. The same is done for the boundary conditions. For the configuration of Fig. 1, where $\bar{J}$ is a fixed current independent of N, it is found that all equations and boundary conditions are satisfied by keeping only odd powers for $\bar{E}$ and even powers for $\bar{H}$ in the expansions. Thus

$$\bar{E} = \bar{E}_1/N + \bar{E}_3/N^3 + \cdots$$
$$\bar{H} = \bar{H}_0 + \bar{H}_2/N^2 + \cdots. \quad (2)$$

Maxwell's equations give, in the dielectric,

$$\text{curl}\,\bar{E}_1 = -jkR_0\bar{H}_0$$
$$\text{curl}\,\bar{H}_0 = \frac{jk}{R_0}\bar{E}_1 + \bar{J} \quad (3)$$

where $R_0 = 120\pi\,\Omega$ is the characteristic resistance of vacuum. We have omitted the equations for the higher order terms, as well as the equations which express that all $\bar{E}_i$ and $\bar{H}_i$ are divergenceless. In vacuo

$$\text{curl}\,\bar{E}_1' = -jkR_0\bar{H}_0'$$
$$\text{curl}\,\bar{H}_0' = 0$$
$$\text{curl}\,\bar{H}_2' = \frac{jk}{R_0}\bar{E}_1'. \quad (4)$$

Notice that primes are used to denote the fields outside the resonator.

B. The Resonant Modes

The boundary conditions on S require all components of the same order to be continuous, the exception being the normal components of $\bar{E}$, which satisfy

$$E_{1n} = 0$$
$$E_{3n} = E_{1n}'. \quad (5)$$

Clearly, the dominant term of $\bar{E}$, i.e., $\bar{E}_1$, is tangent to S on the dielectric side. It therefore satisfies one of the boundary conditions relative to a magnetic wall. The second boundary condition ($\bar{H}$ perpendicular to S) is seldom satisfied, as explained earlier. In the limit $N \to \infty$, the *electric* field $\bar{E}$ is seen, from (2), to approach zero everywhere (while the energy density $\epsilon\,|\bar{E}|^2$ remains finite and different from zero in the dielectric). The *magnetic* field, however, approaches a limit $\bar{H}_0$ different from zero. A resonant mode corresponds to $\bar{J} = 0$ in (3). Its limit magnetic field (which should be denoted by $\bar{H}_{0m}$, but will be written as $\bar{H}_m$ for simplicity) satisfies, from (3),

$$-\,\text{curl curl}\,\bar{H}_m + k_m^2\bar{H}_m = 0, \quad \text{in the dielectric}$$
$$\text{curl}\,\bar{H}_m = 0, \quad \text{in vacuo.} \quad (6)$$

Outside the resonator, $\bar{H}_m$ is therefore irrotational. It decreases at least as fast as $1/R^3$ at large distances. In the following paragraphs, we shall investigate the limit field $\bar{H}_m$ and the eigenvalues k_m^2. It should be emphasized, from the start, that the eigenvectors (6), and their positive eigenvalues k_m^2, are the limit forms for $N \to \infty$. For *finite* N, terms of the kind $\bar{H}_2/N^2$ must be added to $\bar{H}_m$. These terms can be determined by iterative procedures based on equations such as (3) and (4). An important advantage of our method is apparent here. If it is desired to find fields for a series of (high) values of N, it suffices, according to (1), to evaluate a few terms such as $\bar{H}_0, \bar{H}_2$, and to multiply them by the relevant powers of $1/N$. Series (1) then provides the answer.

The electric field, which vanishes for $N \to \infty$, is different from zero for N finite. To first order in $1/N$ it is, from (3),

$$\bar{E} = \frac{\bar{E}_1}{N} = -\frac{jR_0}{Nk_m}\,\text{curl}\,\bar{H}_m, \quad \text{in } V \quad (7)$$

in the resonator. Outside the resonator it is also different from zero, and gives rise to radiated powers and losses. These losses make the eigenvalues in (6) complex, and introduce damping in the otherwise sinusoidal oscillations. The frequency of oscillation of the damped mode f can be obtained from the wavenumber k_m of (6) by the simple formula

$$k_0 = \frac{2\pi f}{c_0} = \frac{1}{N}\,k_m. \quad (8)$$

The time constant of the oscillatory decay is directly related to the Q of the resonance. Expressions for Q will be derived in Sections VI and VII.

The first-order corrections in $1/N$ are sufficient as long as the resonator remains small with respect to λ_0, the wavelength in vacuo. Consider, e.g., the lowest mode of the resonator. Its wavelength in the dielectric is of the order of L. For $\epsilon_r = 100$, $\lambda_0 = N\lambda_{\text{diel}}$ will be of the order of $10L$, and we feel that the low-frequency approximation should be satisfactory in this case. For higher modes, λ_{diel} will be shorter than L, and higher values of ϵ_r are necessary to justify our assumptions. What happens for low values of ϵ_r is not within the province of this article. The evolution of the modes can be followed on the example of the sphere [1], [2], where it appears that the sequence of modes is not conserved, i.e., that the "lowest" mode of (6) does not keep its rank for sufficiently low values of N. "Interior" modes (where the energy is predominantly stored in the sample) and "exterior" modes (where the energy is concentrated on the separation surface and outside the sample) appear at low N. This distinction is not germane to our analysis. We are, indeed, interested in the lowest (finite) eigenvalues of (6), and the eigenvalues of the exterior modes become infinite for $N \to \infty$.

III. CONFINED AND NONCONFINED MODES

The physical meaning of the eigenvectors $\bar{H}_m$ is clear from the preceding section. We shall now concentrate our attention on the *mathematical* properties of the solutions of (6), irrespective of their physical interpretation. Our first task is to *classify* the eigenvectors according to the boundary conditions which they satisfy on S. More precisely, we shall investigate whether boundary surface S can act as a magnetic wall for certain modes. These modes will be called "confined," as their magnetic field vanishes outside the resonator. It is hardly necessary to mention that the confined character only holds in the limit $N \to \infty$, and that all modes have nonzero fields outside the resonator when N is finite.

From the considerations of Section I it follows that the confined modes satisfy

$$-\operatorname{curl}\operatorname{curl}\bar{H}_m + k_m^2\bar{H}_m = 0, \qquad \text{in } V$$
$$\bar{H}_m = 0, \qquad \text{on } S. \tag{9}$$

Clearly, these modes belong to the family of "electric" eigenvectors of an empty cavity V bounded by metallized walls S (boundary condition: zero tangential component on S) [11]. There is, however, an important restrictive feature: the *normal* component of the eigenvector must also vanish everywhere on S. This requirement imposes *three* scalar boundary conditions on S, and therefore overdetermines the problem. We can consequently advance the opinion that the most general resonator has *no* confined modes. This point of view is supported by a study of the modes of the rectangular parallelepiped, which can be written down explicitly, and turn out never to be of the "confined" type [10]. Special symmetries must therefore exist. Cylindrical symmetry (i.e., a waveguide cavity) is not sufficient, as the parallelepiped *is* a waveguide cavity. Let us investigate under which conditions a cylindrical cavity can support a confined mode (Fig. 2). The modes of this cavity belong to two families [11] as follows.

1) Modes with a z component of $\bar{H}_m$ of the form

$$(\bar{H}_m)_z = \cos\frac{n\pi z}{L}\,\phi_m\bar{u}_z. \tag{10}$$

Here, ϕ_m is a Dirichlet eigenfunction of the cross section. Clearly, these modes do not vanish on the end faces, and cannot therefore be confined.

2) Modes which are purely transverse, and for which

$$\bar{H}_m = \sin\frac{n\pi z}{L}\,\bar{u}_z \times \operatorname{grad}\psi_m \tag{11}$$

where ψ_m is a Neumann eigenfunction. Clearly, these modes will be confined if, and only if, both ψ_m and $\partial\psi_m/\partial n$ vanish on the lateral walls. This condition is very special, and is satisfied only by the modes of revolution of a *circular* cylinder. These considerations lead us to assert that the confined modes exist only in bodies of revolution, and that they are ϕ independent (Fig. 3). Their general form is

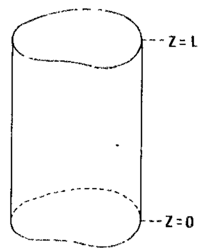

Fig. 2. Waveguide cavity.

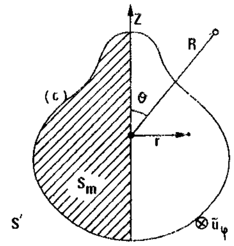

Fig. 3. Resonator of revolution.

$$\bar{H}_m = \beta_m(r,z)\,\bar{u}_\phi \tag{12}$$

where β_m satisfies, from (9),

$$\frac{\partial^2\beta}{\partial r^2} + \frac{1}{r}\frac{\partial\beta}{\partial r} + \frac{\partial^2\beta}{\partial z^2} - \frac{\beta}{r^2} + k_m^2\beta = 0. \tag{13}$$

In addition, β_m vanishes on the axis and on the outer contour (c). For a circular cavity of radius a, for example,

$$\beta_{ps} = \sin\frac{p\pi z}{L}\,J_1\left(x_s\frac{r}{a}\right) \tag{14}$$

with $J_1(x_s) = 0$, and

$$k_{ps}^2 = \left(\frac{p\pi}{L}\right)^2 + \left(\frac{x_s}{a}\right)^2. \tag{15}$$

An exception to the requirement of ϕ independence is afforded by the sphere. The sphere possesses very special symmetry properties, and should therefore support a large number of confined modes. This point of view is confirmed by a detailed analysis of the cavity modes of the sphere [11]. A large number of these have no radial component, and are therefore of the confined type, as their H_n is automatically zero on S. Their ϕ dependence is of the form $\cos m\phi$ or $\sin m\phi$.

IV. ORTHOGONALITY PROPERTIES

A. Orthogonality Properties of the Magnetic Field

Consider two eigenvectors satisfying the differential equations

$$-\operatorname{curl}\operatorname{curl}\bar{H}_m + k_m^2\bar{H}_m = 0$$
$$-\operatorname{curl}\operatorname{curl}\bar{H}_p + k_p^2\bar{H}_p = 0 \tag{16}$$

in the dielectric. Dotting these equations, respectively, with $\bar{H}_p$ and $\bar{H}_m$ gives, after substraction and integration,

$$(k_m^2 - k_p^2) \iiint_V \bar{H}_m \cdot \bar{H}_p \, dV = \iiint_V [-\bar{H}_m \cdot \text{curl curl } \bar{H}_p + \bar{H}_p \cdot \text{curl curl } \bar{H}_m] \, dV. \quad (17)$$

A classical Green's theorem allows us to transform the right-hand member into

$$I = \iint_S [\bar{H}_m \cdot (\text{curl } \bar{H}_p \times \bar{u}_n) - \bar{H}_p \cdot (\text{curl } \bar{H}_m \times \bar{u}_n)] \, dS. \quad (18)$$

The symbols $\bar{H}_m$ and $\bar{H}_p$ stand for the field values along S, just inside the dielectric. But $\bar{H}$ is continuous across S. Outside the dielectric, however, $\bar{H}$ is irrotational, and equal to grad ψ, where ψ is a suitable potential function. In (18), we shall therefore replace $\bar{H}_m$ and $\bar{H}_p$ by $\text{grad}_s \psi_m$ and $\text{grad}_s \psi_p$, where the subscript S refers to derivatives along the surface. Thus

$$I = \iint_S \text{grad}_s \psi_m \cdot (\text{curl } \bar{H}_p \times \bar{u}_n) \, dS - \iint_S \text{grad}_s \psi_p \cdot (\text{curl } \bar{H}_m \times \bar{u}_n) \, dS. \quad (19)$$

These expressions can be simplified by use of the following key relationships [11]:

$$\iint_S \text{grad}_s f \cdot (\bar{u}_n \times \bar{P}) \, dS = -\iint_S f \, \text{div}_s(\bar{u}_n \times \bar{P}) \, dS = \iint_S f \bar{u}_n \cdot \text{curl } \bar{P} \, dS. \quad (20)$$

Applying this transformation to (19) gives, because of (20),

$$I = k_m^2 \iint_S \psi_p (\bar{u}_n \cdot \bar{H}_m) \, dS - k_p^2 \iint_S \psi_m (\bar{u}_n \cdot \bar{H}_p) \, dS = k_m^2 \iint_S \psi_p \frac{\partial \psi_m}{\partial n} \, dS - k_p^2 \iint_S \psi_m \frac{\partial \psi_p}{\partial n} \, dS. \quad (21)$$

The form of the surface integrals suggests use of a Green's theorem, namely,

$$\iiint_{V'} \bar{H}_m \cdot \bar{H}_p \, dV = \iiint_{V'} \text{grad } \psi_m \cdot \text{grad } \psi_p \, dV = -\iiint_{V'} \psi_m \nabla^2 \psi_p \, dV - \iint_S \psi_m \frac{\partial \psi_p}{\partial n} \, dS. \quad (22)$$

The integral over the large sphere at infinity vanishes because ψ is regular at infinity, which implies that ψ approaches zero at least as fast as $1/R$, and $\partial\psi/\partial n$ as fast as $1/R^2$. As ψ is harmonic,

$$\iiint_{V'} \bar{H}_m \cdot \bar{H}_p \, dV = -\iint_S \psi_m \frac{\partial \psi_p}{\partial n} \, dS = -\iint_S \psi_p \frac{\partial \psi_m}{\partial n} \, dS. \quad (23)$$

Taking these properties into account transforms (21) into

$$I = -(k_m^2 - k_p^2) \iiint_{V'} \bar{H}_m \cdot \bar{H}_p \, dV. \quad (24)$$

Combining (17) and (24) leads to the final orthogonality property, valid for eigenvectors belonging to different eigenvalues:

$$\iiint_{V+V'} \bar{H}_m \cdot \bar{H}_p \, dV = 0. \quad (25)$$

It is to be noticed that the proof of the orthogonality is still valid when one (or both) of the eigenvectors is of the "confined" type. For such a case, the integral over V' vanishes automatically, and the resulting integral in (25) is over V only.

The set of eigenvectors $\bar{H}_m$ is not suitable to expand an *arbitrary* vector function defined on V and V'. The set is complete with respect to solenoidal vectors which behave as $\bar{H}_m$ outside V, i.e., which are irrotational in V', and vanish at least as fast as $1/R^3$ at large distances.

B. Orthogonality Properties of the Electric Field

The electric field of a mode is proportional to curl $\bar{H}_m$. The orthogonality proof for curl $\bar{H}_m = \bar{A}_m$ proceeds much as for the magnetic field. We start from equations satisfied by $\bar{A}_m$ and $\bar{A}_p$, viz.,

$$-\text{curl curl } \bar{A}_m + k_m^2 \bar{A}_m = 0$$
$$-\text{curl curl } \bar{A}_p + k_p^2 \bar{A}_n = 0 \quad (26)$$

and manipulate them as in the preceding paragraph. We write

$$I = (k_m^2 - k_p^2) \iiint_V \bar{A}_m \cdot \bar{A}_p \, dV = \iint_S [\bar{A}_m \cdot (\text{curl } \bar{A}_o \times \bar{u}_n) - \bar{A}_p \cdot (\text{curl } \bar{A}_m \times \bar{u}_n)] \, dS.$$

But curl $\bar{A} = \text{curl}(\text{curl } \bar{H}) = k^2 \bar{H}$. It follows that

$$I = k_p^2 \iint_S \bar{H}_p \cdot (\bar{u}_n \times \text{curl } \bar{H}_m) \, dS - k_m^2 \iint_S \bar{H}_m \cdot (\bar{u}_n \times \text{curl } \bar{H}_p) \, dS.$$

The integrals are of the type encountered in (18), and have already been transformed there. Following the steps leading from (18) to (21) gives here

$$I = k_p^2 \left(k_m^2 \iint_S \psi_p \frac{\partial \psi_m}{\partial n} \, dS \right) - k_m^2 \left(k_p^2 \iint_S \psi_m \frac{\partial \psi_p}{\partial n} \, dS \right).$$

This value is zero because of (23). We conclude that, for vectors corresponding to different eigenvalues,

$$\iiint_V \bar{A}_m \cdot \bar{A}_p \, dV = \iiint_V \text{curl } \bar{H}_m \cdot \text{curl } \bar{H}_p \, dV = 0. \quad (27)$$

The orthogonality property is over the dielectric volume, and not over all space (as was the case for $\bar{H}_m$). The curl $\bar{H}_m$ form a complete solenoidal set *in* V.

C. Equipartition of Energy

The electric field, as remarked before, is of order $1/N$. It vanishes when N grows without limit, and so does the electric energy in the vacuum region. The average electric energy in the dielectric does not vanish, however. For a resonant mode it is, from (7),

$$\epsilon = \frac{1}{4}\iiint_V \epsilon_r\epsilon_0 \, |\bar{E}|^2 \, dV = \frac{\epsilon_0}{4}\iiint_V |\bar{E}_1|^2 \, dV$$

$$= \frac{\epsilon_0}{4}\frac{R_0^2}{k_m^2}\iiint_V |\operatorname{curl} \bar{H}_m|^2 \, dV$$

$$= \frac{\mu_0}{4k_m^2}\iiint_V |\operatorname{curl} \bar{H}_m|^2 \, dV.$$

By applying methods similar to those utilized in the proof of the orthogonality property, one easily shows that

$$\epsilon = \frac{\mu_0}{4}\iiint_{V+V'} |\bar{H}_m|^2 \, dV = \frac{\epsilon_r\epsilon_0}{4}\iiint_V |\bar{E}|^2 \, dV. \quad (28)$$

The electric energy is therefore equal to the magnetic energy. The familiar property of equipartition of energy in a resonant mode is seen to be still valid in the present case.

V. VARIATIONAL PRINCIPLES

A. Variational Principle for Nonconfined Modes

We shall give this variational principle in a form which is well suited for the application of the method of finite elements. Let grad ψ_m represent $\bar{H}_m$ outside the resonator. The desired functional is

$$J(\psi,\bar{H}) = -\iiint_{V'} |\operatorname{grad}\psi|^2 \, dV + \iiint_V \left\{\frac{1}{k^2}|\operatorname{curl}\bar{H}|^2 - |\bar{H}|^2\right\} dV. \quad (29)$$

To derive the Euler equations, we assume that J is stationary about ψ_0 and $\bar{H}_0$, and write

$$\psi = \psi_0 + \epsilon\eta$$
$$\bar{H} = \bar{H}_0 + \epsilon\bar{\mu}.$$

We insert these values in (29), and obtain, for J, a polynomial of the second degree in the small parameter ϵ. J is stationary when $\partial J/\partial\epsilon$ vanishes for $\epsilon = 0$, i.e., when the coefficient of ϵ in the polynomial vanishes. This condition yields

$$-\iiint_{V'} \operatorname{grad}\psi_0\cdot\operatorname{grad}\eta \, dV + \frac{1}{k^2}\iiint_V \operatorname{curl}\bar{H}_0\cdot\operatorname{curl}\bar{\mu} \, dV - \iiint_V \bar{H}_0\cdot\bar{\mu} \, dV = 0.$$

We apply well-known vector relationships to introduce η and $\bar{\mu}$ everywhere as coefficients in the integrand. The preceding condition becomes

$$\iiint_{V'} \eta\nabla^2\psi_0 \, dV + \iint_S \eta\frac{\partial\psi_0}{\partial n} \, dS + \frac{1}{k^2}\iint_S \bar{\mu}\cdot(\operatorname{curl}\bar{H}_0 \times \bar{u}_n) \, dS + \iiint_V \bar{\mu}\cdot\left[\frac{1}{k^2}\operatorname{curl}\operatorname{curl}\bar{H}_0 - \bar{H}_0\right] dV = 0. \quad (30)$$

The contribution from the large sphere at infinity is ignored because the test functions must be regular at infinity, which means that both ψ_0 and η are regular there. The form of (30) indicates that the stationarizing functions satisfy the Euler equations

$$\nabla^2\psi_0 = 0$$

$$\frac{1}{k^2}\operatorname{curl}\operatorname{curl}\bar{H}_0 - \bar{H}_0 = 0.$$

These are precisely the differential equations satisfied by the sought eigenvectors. To transform the surface integrals in (30), let us choose the test functions such that the tangential components of the eigenvectors are continuous on S. More precisely, let

$$\operatorname{grad}_s \psi = (\bar{H})_{\tan}, \qquad \text{on } S$$
$$\operatorname{grad}_s \eta = (\bar{\mu})_{\tan}, \qquad \text{on } S. \quad (31)$$

From (20), the second surface integral in (30) can be rewritten as

$$\frac{1}{k^2}\iint \operatorname{grad}_s\eta\cdot(\operatorname{curl}\bar{H}_0 \times \bar{u}_n) \, dS = -\frac{1}{k^2}\iint_s \eta\bar{u}_n\cdot\operatorname{curl}\operatorname{curl}\bar{H}_0 \, dS = -\iint_s \eta(\bar{u}_n\cdot\bar{H}_0) \, dS.$$

The sum of the surface integrals is

$$\iint_s \eta\left[\frac{\partial\psi_0}{\partial n} - \bar{u}_n\cdot\bar{H}_0\right] dS.$$

A stationary field therefore satisfies the natural boundary condition

$$\frac{\partial\psi_m}{\partial n} = \bar{u}_n\cdot\bar{H}_m, \qquad \text{on } S. \quad (32)$$

B. Variational Principle for Modes of Revolution

In a volume of revolution, the nonconfined modes of revolution are of the type (Fig. 3)

$$\bar{H}_m = \operatorname{curl}(\alpha\bar{u}_\phi) = \frac{1}{r}\operatorname{grad}(r\alpha) \times \bar{u}_\phi, \qquad \text{in } V$$

$$\bar{H}_m = \operatorname{curl}(\gamma\bar{u}_\phi) = \frac{1}{r}\operatorname{grad}(r\gamma) \times \bar{u}_\phi, \qquad \text{in } V'. \quad (33)$$

The functions α and γ satisfy the differential equations

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\alpha)\right]+\frac{\partial^2\alpha}{\partial z^2}+k^2\alpha$$

$$=\frac{\partial^2\alpha}{\partial r^2}+\frac{1}{r}\frac{\partial\alpha}{\partial r}-\frac{\alpha}{r^2}+\frac{\partial^2\alpha}{\partial z^2}+k^2\alpha=0, \qquad \text{in } V$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\gamma)\right]+\frac{\partial^2\gamma}{\partial z^2}=0, \qquad \text{in } V' \tag{34}$$

with the boundary conditions

1) α and γ zero on the axis
2) $\alpha = \gamma$ and $\partial\alpha/\partial n = \partial\gamma/\partial n$ on contour (c)
3) γ proportional with $\sin\theta/R^2$ at large distances. (35)

Solution of (34) and (35) proceeds by any of the methods available for the determination of eigenvalues and eigenfunctions. If it is desired to apply the method of finite elements, a suitable functional,[1] derived from (29), is

$$J(\alpha,\gamma) = \iint_S \left[\frac{1}{r}\operatorname{grad}(r\gamma)\right]^2 r\,dS + \iint_{S_m}\left\{\left[\frac{1}{r}\operatorname{grad}(r\alpha)\right]^2 - k^2\alpha^2\right\} r\,dS \tag{36}$$

where S_m is the meridian cross section, and S the boundary surface.

C. *Variational Principle for the Confined Modes*

The variational principle for a confined mode is classical, as this mode is one of the "electric" eigenvectors of the volume of revolution [11]. The eigenvector is of the form $\bar{H}_m = \beta\bar{u}_\phi$, and the differential equation satisfied by β is given in (13). The relevant functional is

$$J = \iiint_V \left\{|\operatorname{curl}\bar{H}|^2 - k^2|\bar{H}|^2\right\} dV$$

$$= \iint_S \left\{\left(\frac{\partial\beta}{\partial z}\right)^2 + \frac{1}{r^2}\left[\frac{\partial}{\partial r}(r\beta)\right]^2 - k^2\beta^2\right\} r\,dS. \tag{37}$$

The test functions must vanish on the contour and on the axis.

VI. THE Q OF THE NONCONFINED MODES

A. *Magnetic Moment of the Modes*

1) When N is infinite, the electric field vanishes outside the resonator. No energy is radiated, and the Q is infinite. The calculation of the Q, therefore, has meaning only if N is finite. Q is given by the classical formula

$$Q = \frac{\omega \times \text{reactive energy}}{\text{average dissipated power}}. \tag{38}$$

We shall only consider losses due to radiation, and neglect, as mentioned in the introduction, the influence of material losses. These introduce a Q_{diel}, and the total $1/Q$ is the sum of $1/Q_{\text{rad}}$ and $1/Q_{\text{diel}}$. Present-day materials have Q_{diel} of the order of 1000 at X band (for N's of the order of 10).

To evaluate Q for large N, we shall replace the various factors in (38) by their dominant term in an expansion in $1/N$. The resonant angular frequency ω, for example, will be written as $k_m(c/N)$, and the reactive energy set equal to the value which obtains for $N \to \infty$ (the unperturbed value). This energy is the sum of the average electric and magnetic energies in the mode, i.e., twice the value appearing in (28). The main problem is the calculation of the average radiated power. To determine the radiated field, we replace the resonator by its polarization currents [3], [8], [12], which are

$$\bar{J} = j\omega\epsilon_0(N^2-1)\bar{E}.$$

For very high values of N,

$$\bar{J} \approx j\omega\epsilon_0 N^2\bar{E} = \operatorname{curl}\bar{H}_m. \tag{39}$$

It is this value which should be inserted in the expression for the vector potential at large distances, viz.,

$$\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi}\iiint_V \bar{J}(\bar{r}')\frac{\exp[-j(k/N)\,|\bar{r}-\bar{r}'|]}{|\bar{r}-\bar{r}'|}\,dV'$$

$$= \frac{\mu_0}{4\pi}\frac{\exp[-j(k/N)R]}{R}\left[\iiint_V \bar{J}\,dV + j\frac{k}{N}\iiint_V(\bar{u}\cdot\bar{r}')J\,dV - \frac{k^2}{2N^2}\iiint_V(\bar{u}\cdot\bar{r}')^2\bar{J}\,dV + \cdots\right] \tag{40}$$

where $\bar{u}$ is the direction of observation. The first integral vanishes, as

$$\iiint_V \bar{J}\,dV = \iiint_V \operatorname{curl}\bar{H}_m\,dV = \iint_S(\bar{u}_n\times\bar{H}_m)\,dS$$

$$= \iint_S \bar{u}_n \times \operatorname{grad}_s\psi_m\,dS = 0. \tag{41}$$

The last integral is, indeed, zero for a closed surface [11]. The dominant term in (40) is therefore the term in jk/N, which is known to give rise to magnetic dipole and electric quadrupole fields [13]. The electric quadrupole term is zero in our case because volume and surface polarization charges are absent. The volume charges, proportional with $\operatorname{div}\bar{J}$, vanish because $\operatorname{div}\bar{J} = \operatorname{div}(\operatorname{curl}\bar{H}_m) = 0$. The surface charges are also zero because $\bar{J}$ is tangent to S, from (5), (7), and (39). We conclude that the dielectric resonator radiates like a magnetic dipole of moment

$$\bar{p}_m = \frac{1}{2}\iiint_V \bar{r}\times\bar{J}\,dV = \frac{1}{2}\iiint_V \bar{r}\times\operatorname{curl}\bar{H}_m\,dV. \tag{42}$$

This dipole moment is also the dipole moment of the "magnetostatic" field $\bar{H}_m$ surrounding the resonator. Ap-

[1] The application of the method of finite elements to simple shapes such as the coaxial ring or the circular cylinder is in progress. The results will be published in a forthcoming report.

plication of the formula for Q now gives

$$Q = \frac{6\pi N^3}{k_m^5} \frac{\iiint_V |\operatorname{curl} \bar{H}_m|^2 \, dV}{|\bar{p}_m|^2}$$

$$= \frac{24\pi N^3}{k_m^5} \frac{\iiint_V |\operatorname{curl} \bar{H}_m|^2 \, dV}{\left| \iiint_V \bar{r} \times \operatorname{curl} \bar{H}_m \, dV \right|^2}. \tag{43}$$

It is seen that the quality factor is proportional with N^3, an interesting result indeed.

2) The value of $\bar{p}_m$ appearing in (42) involves curl $\bar{\boldsymbol{H}}_m$. It is possible to derive an equivalent formula in terms of $\bar{\boldsymbol{H}}_m$ by splitting $\bar{p}_m$ in three terms of the form

$$I = \tfrac{1}{2}\bar{u}_x \times \iiint_V x \operatorname{curl} \bar{H}_m \, dV = \tfrac{1}{2}\bar{u}_x \times \left[\iiint_V \operatorname{curl} (x\bar{H}_m) \, dV - \iiint_V \operatorname{grad} x \times \bar{H}_m \, dV \right]. \tag{44}$$

Further manipulation of this expression leads to

$$\bar{p}_m = \iiint_V \bar{H}_m \, dV - \iint_S \psi_m \bar{u}_n \, dS \tag{45}$$

where ψ_m is the potential from which $\bar{H}_m$ can be derived outside the resonator. The new value of the quality factor is

$$Q = \frac{6\pi N^3}{k_m^3} \frac{\iiint_{V+V'} |\bar{H}_m|^2 \, dV}{\left[\iiint_V \bar{H}_m \, dV - \iint_S \psi_m \bar{u}_n \, dS \right]^2}. \tag{46}$$

B. Application to Modes of Revolution

For the modes of revolution appearing in (33), formulas (42) and (43) are particularly appropriate, as curl $\bar{H}_m$ has the simple form

$$\operatorname{curl} \bar{H}_m = \operatorname{curl} \operatorname{curl} (\alpha \bar{u}_\phi) = k_m^2 \alpha \bar{u}_\phi \tag{47}$$

in the dielectric. The quantities of interest become

$$\bar{p}_m = \pi k_m^2 \iint_{S_m} \alpha r^2 \, dS \bar{u}_z \tag{48}$$

and

$$Q = \frac{12 N^3}{k_m^5} \frac{\iint_{S_m} \alpha^2 r \, dS}{\left[\iint_{S_m} \alpha r^2 \, dS \right]^2}. \tag{49}$$

The lines of force of these modes are circular (azimuthal) for $\bar{E}$, and meridian for $\bar{H}$. Outside the resonator, the lines of $\bar{H}$ have the general aspect of those of a magnetic dipole.

C. Higher Modes

A resonant mode will radiate like a magnetic dipole only if $\bar{p}_m$ is different from zero. For certain modes this moment vanishes. A particular example, chosen among many others, is the following mode of a body of revolution:

$$\operatorname{curl} \bar{H}_m = \sin 2\phi \bar{A}_m(r,z) + \cos 2\phi B_m(r,z) \bar{u}_\phi. \tag{50}$$

In this expression $\bar{A}_m$ is a suitable meridian vector. For such modes, the term in jk/N in the expansion (40) for the vector potential vanishes, and the first term to take into consideration is the term in $1/N^2$. The far fields are now of order $1/N^3$ (instead of $1/N^2$), and the radiated power is of order $1/N^6$. As a result, Q is proportional to N^5. More generally, an additional factor of N^2 appears in the quality factor Q each time a leading term disappears in the expansion for the vector potential. The Q of the corresponding mode obviously grows faster with N than was the case for the dipole. The phenomenon can be followed clearly on the multipole modes of the sphere, which have been studied in great detail [1], [2]. The high value of Q means that little energy is radiated, hence that boundary surface S is almost "leakproof." This remark explains why the "magnetic wall" condition is a better approximation for higher modes than for lower ones. Our analysis confirms the conclusions of Yee [5], [6]. This author introduces E modes and H modes. The H mode is defined as the mode which has a large normal component of magnetic field at the boundary surface. It corresponds to our nonconfined mode. The E mode is a mode with no predominant normal component of $\bar{H}$ at the surface. It corresponds to a confined mode. Yee compares a spherical dielectric resonator of high permittivity and a spherical resonator of the same size and permittivity, bounded by an open-circuit surface. For $\epsilon_r = 100$, the resonant frequencies of the E modes agree to within 1 percent. The lower H modes, however, evidence differences of up to 14 percent.

Evaluation of Q for the quadrupole, octupole, etc., modes is very intricate. The calculations will be carried out for the confined modes only, which are precisely of the type $\bar{p}_m = 0$. Here, the evaluation of Q can serve as an example for analog calculations concerning the higher modes.

VII. THE Q OF THE CONFINED MODES

A. Electric Moment of the Modes

1) For the modes described by (12) the polarization currents $\bar{J}$ are meridian vectors independent of ϕ. For such currents,

$$\bar{p}_m = \frac{1}{2} \iiint_V \bar{r} \times \bar{J} \, dV = 0$$

automatically. The next term in the expansion for the vector potential is

$$\bar{A}_1 = -\frac{\mu_0}{8\pi}\frac{k^2}{N^2}\frac{\exp[-j(kR/N)]}{R}\iiint_V (\bar{u}\cdot\bar{r}')^2\bar{J}\,dV' + \cdots \tag{51}$$

in which we set $\bar{J} = \text{curl}\,\bar{H}_m$. There is, however, a second contribution of order $1/N^2$. This contribution is obtained by replacing the value of $\bar{J}$ appearing in (39) by an expression including higher terms. Thus

$$\bar{J} = \text{curl}\,\bar{H} - j\omega\epsilon_0\bar{E} = j\omega\epsilon_0(N^2-1)\bar{E} = \frac{N^2-1}{N^2}\,\text{curl}\,\bar{H}$$

$$= \frac{N^2-1}{N^2}\left(\text{curl}\,\bar{H}_m + \frac{1}{N^2}\,\text{curl}\,\bar{H}_2\right)$$

$$= \text{curl}\,\bar{H}_m\left(1-\frac{1}{N^2}\right) + \frac{1}{N^2}\,\text{curl}\,\bar{H}_2. \tag{52}$$

The second contribution in $1/N^2$ stems from the insertion of $(1/N^2)$ curl $\bar{H}_2$ into (40). This gives

$$\bar{A}_2(\bar{r}) = \frac{\mu_0}{4\pi}\frac{\exp[-j(kR/N)]}{RN^2}\iiint_V \text{curl}\,\bar{H}_2\,dV. \tag{53}$$

2) To evaluate $\bar{A}_1$, we insert

$$\bar{J} = \text{curl}\,\bar{H}_m = \text{curl}\,(\beta\bar{u}_\phi) = (1/r)\,\text{grad}\,(r\beta)\times\bar{u}_\phi$$

$$= \frac{\partial\beta}{\partial z}\bar{u}_r + \frac{1}{r}\frac{\partial}{\partial r}(r\beta)\bar{u}_z \tag{54}$$

in (51), where $\bar{u}$ is a unit vector characterizing the direction of observation (θ,ϕ). Elementary calculations yield, taking into account that β vanishes on (c) (Fig. 3),

$$\iiint_V (\bar{u}\cdot\bar{r}')^2\bar{J}\,dV' = \pi\sin^2\theta\iint_{S_m} r^2(\partial/\partial r)(r\beta)\,dS\bar{u}_z + 2\pi\sin\theta\cos\theta\iint_{S_m}\beta r^2\,dS\bar{u}_r.$$

A few additional manipulations lead to

$$\bar{A}_1 = -\frac{\mu_0}{4\pi}\frac{k^2}{N^2}\frac{\exp[-j(kR/N)]}{R}\sin\theta\iint_{S_m}\beta r^2\,dS\bar{u}_\theta. \tag{55}$$

This vector potential generates an electric dipole field.

3) To evaluate $\bar{A}_2$, we transform the integral in (53) as follows:

$$\bar{I} = \iiint_V \text{curl}\,\bar{H}_2\,dV = \iint_S (\bar{u}_n\times\bar{H}_2)\,dS$$

$$= \iint_S (\bar{u}_n\times\bar{H}_2')\,dS$$

where S is the boundary surface of the volume of revolution. Consider the x component of this integral. Utilizing (4) and (20) gives

$$I_x = \iint_S \bar{u}_x\cdot(\bar{u}_n\times\bar{H}_2')\,dS = \iint_S \text{grad}\,x\cdot(\bar{u}_n\times\bar{H}_2')\,dS$$

$$I_x = -\iint_S x\,\text{div}_s\,(\bar{u}_n\times\bar{H}_2')\,dS = \iint_S x\,\bar{u}_n\cdot\text{curl}\,\bar{H}_2'\,dS$$

$$= \iint_S xj\frac{k}{R_0}\bar{u}_n\cdot\bar{E}_1'\,dS.$$

We conclude that

$$\bar{I} = \iiint_V \text{curl}\,\bar{H}_2\,dV = j\frac{k}{R_0}\iint_S E_{1n}'\bar{r}\,dS$$

$$= j\frac{k}{R_0}\bar{u}_z\iint_S E_{1n}'z\,dS. \tag{56}$$

Inserting this expression in (53) shows that $\bar{A}_2$ also produces an electric dipole field. The total moment, sum of the contributions of $\bar{A}_1$ and $\bar{A}_2$, is

$$\bar{p}_e = \left[-j\frac{\pi k}{Nc_0}\iint_{S_m}\beta r^2\,dS + \frac{\epsilon_0}{N}\iint_S E_{1n}'z\,dS\right]\bar{u}_z. \tag{57}$$

The quality factor can now be written as

$$Q = \frac{6\pi N^3}{k^3c_0^2}\frac{\iiint_V |\bar{H}_m|^2\,dV}{|\bar{p}_e|^2}$$

$$= N^5\frac{12\iint_{S_m}\beta^2 r\,dS}{k^3\left|(1/\pi R_0)\iint_S E_{1n}'z\,dS - jk\iint_{S_m}\beta r^2\,dS\right|^2}. \tag{58}$$

B. Discussion of the Dipole Moment

1) Formulas (57) and (58) indicate that a knowledge of the normal component $E_n' = E_{1n}'/N$ on S is necessary for the evaluation of $\bar{p}_e$ and Q. The determination of E_{1n}' is a potential problem. $\bar{E}_1'$ is indeed irrotational outside the resonator, as $\bar{H}_0'$ vanishes there (4). We therefore set

$$\bar{E}_1' = \text{grad}\,\phi, \qquad \text{in } V'. \tag{59}$$

The tangential component of this vector is known along S. It is, from (7) and (12),

$$(\bar{E}_1')_{\tan} = \frac{R_0}{jk}\frac{1}{r}\,\text{grad}\,(r\beta)\times\bar{u}_\phi = \frac{R_0}{jk}\frac{\partial\beta}{\partial n}\bar{u}_c. \tag{60}$$

As $\bar{E}_1'$ is divergenceless, ϕ must be harmonic. The determination of $E_{1n}' = \partial\phi/\partial n$ therefore requires solution of the problem

$$\nabla^2\phi = 0, \qquad \text{in } V'$$

$$\text{grad}_s\,\phi, \qquad \text{given on } S$$

$$\iint_s E_{1n}'\,dS = \iint \frac{\partial\phi}{\partial n}\,dS = 0. \tag{61}$$

The last condition expresses the absence of total electric charge on the resonator. Notice that $\bar{E}_1'$ has all the earmarks of an electrostatic dipole field, and is therefore of order $1/R^3$ at large distances. Trivial calculations show that the generating moment of this electrostatic field is precisely (57).

It is apparent, from the previous considerations, that the determination of $\bar{p}_e$ (and therefore of Q) is an exterior potential problem. The dipole moment can be obtained from a knowledge of $E_{1n}' = \partial\phi/\partial n$ along S, or from the value of ϕ at large distances. Actual solution proceeds by the classical methods of electrostatics, i.e., by difference equations, finite-element methods associated with a variational principle, or by solution of an integral equation on contour (c). From the knowledge of ϕ the lines of force of $\bar{E}$ can be traced outside the resonator. These lines are meridian, and are similar to those of an electric dipole. The lines of force of $\bar{H}$ are circular (azimuthal).

2) The dipole moment can also be evaluated by use of a formula which gives the fields outside S in terms of certain field components on S, viz., [11]

$$4\pi\bar{H} = -\,\text{grad}\iint_S H_n \frac{\exp\left[-j(k/N)\,|\bar{r}-\bar{r}'|\right]}{|\bar{r}-\bar{r}'|}\,dS' + \frac{jk}{NR_0}\iint_S (\bar{u}_n\times\bar{E})\frac{\exp\left[-j(k/N)\,|\bar{r}-\bar{r}'|\right]}{|\bar{r}-\bar{r}'|}\,dS' + \text{curl}\iint_S (\bar{u}_n\times\bar{H})\frac{\exp\left[-j(k/N)\,|\bar{r}-\bar{r}'|\right]}{|\bar{r}-\bar{r}'|}\,dS'.$$

The far-field version of this formula shows, after a few calculations, that the resonator radiates like a dipole of moment

$$\bar{p}_e = \frac{\epsilon_0}{N}\left[\iint_S E_{1n}\bar{r}'\,dS' - \iint_S \phi\bar{u}_n\,dS\right]. \tag{62}$$

APPENDIX

1) Let us apply our results to the modes of the sphere, for which exact data are available. A typical nonconfined mode is of the form (Fig. 4)

$$\bar{H}_m = 2\cos\theta\left(\frac{\sin kR}{R^3} - k\frac{\cos kR}{R^2}\right)\bar{u}_R + \sin\theta\left(\frac{\sin kR}{R^3} - k^2\frac{\sin kR}{R} - k\frac{\cos kR}{R^2}\right)\bar{u}_\theta \tag{63}$$

in the dielectric, and

$$\bar{H}_m = ka\cos ka\,\text{grad}\left(\frac{\cos\theta}{R^2}\right) = -\frac{2ka\cos ka}{R^3}\cos\theta\bar{u}_R - \frac{ka\cos ka}{R^3}\sin\theta\bar{u}_\theta \tag{64}$$

outside the dielectric. The resonant wavenumber is given by the condition $\sin ka = 0$. Application of (48) and (49), where

$$\alpha = \sin\theta\left(\frac{\sin kR}{R^2} - k\frac{\cos kR}{R}\right) \tag{65}$$

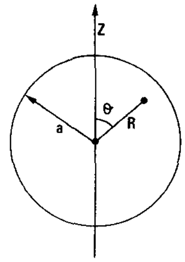

Fig. 4. Spherical resonator.

TABLE I

	TE$_{101}$				TE$_{102}$			
	Q_{ex}	Q_{pert}	$(ka)_{ex}$	$2a/\lambda_0$	Q_{ex}	Q_{pert}	$(ka)_{ex}$	$2a/\lambda_0$
ϵ_r = 14	13	8.35	3	0.254	15	4.20	6.16	0.52
40	50	40	3.08	0.153	39	20	6.20	0.312
86	141	127.5	3.11	0.106	92	64	6.23	0.214

gives

$$\bar{p}_m = -4\pi ka\cos ka\bar{u}_z$$

$$Q = \frac{N^3}{2ka}. \tag{66}$$

The solution $ka = \pi$ yields the TE_{101} mode of Gastine *et al.* [1], [2] and $ka = 2\pi$ the TE$_{102}$ mode. Exact values of Q (termed Q_{ex}) have kindly been communicated to the author by L. Courtois. We compare these to the values given by (66), which will be denoted by Q_{pert}. The values $ka = \pi$ and 2π correspond, as discussed in Section II, to the limit $\epsilon_r \to \infty$. The exact values of ka are quoted in Table I. We have also calculated the ratio of the diameter $2a$ to the wavelength in free space, i.e., $2a/\lambda_0 = (ka)_{ex}/\pi N$. Clearly, the accuracy of Q increases when the diameter of the sphere becomes smaller with respect to the wavelength. Reasonable accuracies (e.g., 10 percent) require a ratio $2a/\lambda_0$ of the order of 0.1 or less.

2) As typical confined mode we take

$$\bar{H}_m = \beta\bar{u}_\phi = \sin\theta\left(\frac{\sin kR}{R^2} - k\frac{\cos kR}{R}\right)\bar{u}_\phi$$

$$\text{curl}\,\bar{H}_m = 2\cos\theta\bar{u}_R\left(\frac{\sin kR}{R^3} - \frac{k\cos kR}{R^2}\right) + \sin\theta\bar{u}_\theta\left(\frac{\sin kR}{R^3} - \frac{k\cos kR}{R^2} - k\frac{2\sin kR}{R}\right). \tag{67}$$

The resonance condition is

$$\sin ka - ka\cos ka = 0. \tag{68}$$

For this mode

$$E_{n1}' = 2jR_0k^2\cos ka\cos\theta. \tag{69}$$

Application of (57) and (58) gives, after trivial calculations,

TABLE II

	TM_{101}				TM_{102}			
	Q_{ex}	Q_{pert}	$(ka)_{ex}$	$2a/\lambda_0$	Q_{ex}	Q_{pert}	$(ka)_{ex}$	$2a/\lambda_0$
ϵ_r = 14	6	4.1	4.21	0.358	11.3	7.9	7.69	0.655
50	84	97	4.38	0.210	23.7	19	7.59	0.364
86	330	375	4.43	0.152	62.5	74	7.61	0.261

$$\bar{p}_e = j(4\pi/Nc)ka^2 \sin ka\bar{u}_z$$

$$Q = N^5/2k_m{}^3a^3. \tag{70}$$

The values $ka = 4.49$ and $ka = 7.73$ yield, respectively, the TM_{101} and TM_{102} modes of Gastine, for which the results shown in Table II hold.

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The Excitation of Dielectric Resonators of Very High Permittivity

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***Abstract*—The response of a dielectric resonator excited by either interior volume sources or incident exterior waves is investigated. Special attention is devoted to phenomena at resonance, and in particular to the induced electric and magnetic dipoles. Simple formulas are obtained for the scattering cross section. The material of the resonator is assumed lossless and of very high permittivity.**

I. INTRODUCTION

IN A PRECEDING article [1] we have investigated the nature and properties of the modes of a dielectric resonator of very high permittivity. In the present paper we make use of the modal properties, and in particular of the orthogonality relationships, to investigate the excitation of a resonator by interior volume sources or, more realistically, by exterior incident fields. Our general method of attack is to assume that the index of refraction N of the (lossless) dielectric is large, and to expand the fields as

$$\bar{E} = \bar{E}_0 + \frac{\bar{E}_1}{N} + \frac{\bar{E}_2}{N^2} + \cdots$$

$$\bar{H} = \bar{H}_0 + \frac{\bar{H}_1}{N} + \frac{\bar{H}_2}{N^2} + \cdots. \tag{1}$$

These expansions are inserted in Maxwell's equations, and terms of equal orders on both sides of these equations are equated. The mechanics of the procedure will be described in subsequent paragraphs. Our main purpose is to determine the dominant terms in (1), and in particular the behavior of these terms in the vicinity of a resonance $k = k_m$. In the limit $N \to \infty$, the magnetic field $\bar{H}_0$ near resonance must be proportional to the relevant eigenvector $\bar{H}_m$, solution of [1],

$$-\text{curl curl } \bar{H}_m + k_m{}^2\bar{H}_m = 0 \qquad \text{in } V$$

$$\text{curl } \bar{H}_m = 0 \qquad \text{in } V'. \tag{2}$$

These eigenvectors satisfy the important orthogonality properties [1]

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